METRIC AND TOPOLOGICAL SPACES: EXAM 2022/23

A. V. KISELEV

Problem 1 (each new full proof counts 10%)**.** Let $(\mathcal{X}, \mathrm{d}_{\mathcal{X}})$ be a metric space, let $\mathcal{Y} = \{A \subseteq \mathcal{X} \mid A \neq \varnothing\}$, and for any $U, V \in \mathcal{Y}$ put $\varrho(U, V) = \inf_{x \in U, y \in V} \mathrm{d}_{\mathcal{X}}(x, y)$. Is ϱ always a metric on $\mathcal{Y}$?
(state and prove, e.g., by counterexample)

Problem 2 (15%)**.** If $\mathcal{X}$ is Hausdorff and $x \in \mathcal{X}$, then the intersection of all open subsets of $\mathcal{X}$ containing x is the singleton set $\{x\}$.
(prove)

Problem 3 (15%)**.** Give an example of non-empty topological space containing more than one point and with co-finite topology which is Hausdorff.

Problem 4 (20%)**.** If $\mathcal{X}$ is a connected space containing more than one point, and if $\{x\}$ is closed for every $x \in \mathcal{X}$, then the number of points in $\mathcal{X}$ is infinite. (prove)

Problem 5 (20%)**.** Prove: $\mathbb{R}^2$ with co-finite topology is compact.

Problem 6 (20%)**.** Let $\mathcal{X}$ be a complete metric space and $f\colon \mathcal{X} \to \mathcal{X}$ be a mapping such that for some $r > 1$ its r-time iteration $(f\circ)^r = f \circ \ldots \circ f$ is a Banach contraction.

Prove that f has a unique fixed point p in $\mathcal{X}$, regardless of f itself being (or maybe not?) a contraction.

Date: November 11, 2022. Good luck & take care !